Quantum Gravity via Causal Dynamical Triangulations

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Abstract

"Causal Dynamical Triangulations" (CDT) represent a lattice regularization of the sum over spacetime histories, providing us with a non-perturbative formulation of quantum gravity. The ultraviolet fixed points of the lattice theory can be used to define a continuum quantum field theory, potentially making contact with quantum gravity defined via asymptotic safety. We describe the formalism of CDT, its phase diagram, and the *quantum geometries* emerging from it. We also argue that the formalism should be able to describe a more general class of quantum-gravitational models of Hořava-Lifshitz type.

1 Introduction

At this stage, there is no certainty how to best reconcile the classical theory of relativity with quantum mechanics. Applying the well-tested methods of quantization to gravity – defined by the Einstein-Hilbert action – and quantizing the fluctuations around a classical solution to Einstein's equations leads to a non-renormalizable theory. This happens because in four spacetime dimensions the mass dimension of the gravitational coupling constant G (in units where $\hbar$ and c are 1) is -2, whereas it should be larger than or equal to 0 for the theory to be renormalizable perturbatively. One would therefore expect the perturbative effective quantum field theory description to break down at energies E satisfying $GE^2 \gtrsim 1$.

There are of course well-known examples where the non-renormalizability of a quantum field theory in the ultraviolet (UV) was eventually resolved by introducing new degrees of freedom, missed initially because they were not directly observable at low energies. The electroweak theory is an example where perturbative renormalizability was "regained" in this way. The theory was described first by a four-fermion interaction with an associated Fermi coupling G_F of mass dimension -2, just like the Newton constant G in gravity. As a result, its perturbation theory breaks down at energies with $G_F E^2 \gtrsim 1$. However, it turns out that for energies above $1/\sqrt{G_F} \approx M_W$, the mass of the W-particle, the four-fermion theory has to be replaced by the $SU(2)$-gauge theory of the weak interactions, which contains new excitations, the W- and Z-bosons. The new electroweak theory *is* a renormalizable quantum field theory.

Similarly, in the 1960s the low-energy scattering of pions was described by a non-linear sigma model, another non-renormalizable quantum field theory whose coupling constant, the pion decay constant F_π-squared, has mass dimension -2. However, high-energy scattering at energies beyond $1/F_\pi$ is no longer described well by the non-linear sigma model, because it starts probing the intrinsic structure of the pions. A correct description has to incorporate appropriate new degrees of freedom, the quarks and gluons, and the corresponding quantum theory – quantum chromodynamics – is perfectly renormalizable.

There is no obvious reason which prevents us from writing down a perturbative (and non-renormalizable) expansion for gravity around some classical background geometry, say, flat Minkowski spacetime, if we are interested in an effective quantum field-theoretic description whose range of applicability does not extend beyond energies with $GE^2 \approx 1$. In view of the examples cited above, it is then tempting to conjecture that the apparent non-renormalizability of gravity could be resolved by the appearance of new degrees of freedom at higher energies, rendering the theory renormalizable after all.

A solution of this kind may be in the form of a superstring theory in a higher-dimensional spacetime, where the gravitational excitations are intertwined with infinitely many new degrees of freedom in such a way as to cure the UV problem. Although string theory cannot be ruled out as the correct answer, the world picture

it provides has yet to be verified. In particular, supersymmetry – predicted by string theory – has not yet been observed at the Large Hadron Collider. Of course, even if no evidence of supersymmetry is found at this or future colliders, it may still be present at even higher energies. In this sense, the absence of observational evidence for supersymmetry does not disprove superstring theory as such, although it makes it less compelling as a resolution of the problem of unifying gravity and quantum theory.

There are other potential resolutions to the problem of finding a suitable "ultraviolet completion" of perturbative quantum gravity, which are not based on fundamental, string-like excitations and do not obviously require the existence of supersymmetry or extra dimensions. These are so-called non-perturbative approaches, whose starting point typically consists of a set of dynamical degrees of freedom closely modeled on those of classical gravity ("curved geometry" in one way or other), together with a non-perturbative prescription for quantization. A concrete example, that of *Causal Dynamical Triangulations*, will be described in some detail below. Its geometric degrees of freedom, in presence of a UV cut-off, are given in terms of triangulated, piecewise flat spacetimes with discrete curvature assignments. Its non-perturbative quantization follows that of a standard lattice field theory, albeit with a dynamical rather than a fixed lattice.

An obvious charm of such a purely quantum field-theoretic ansatz lies in its minimalism, and the absence – to a large degree – of free parameters and other "tunable" ingredients. On the other hand, a key difficulty of this type of approach is to demonstrate that it is related to classical gravity in a suitable limit, something that is not at all obvious once one has moved beyond linearized quantum fields on a fixed background spacetime. One also needs to spell out what it means for the non-perturbative theory to *exist*, which likewise is non-trivial in a background-free description where "observables" are hard to come by.

In parallel with advances in string theory, also research in the wider area of non-perturbative quantum gravity has seen a steady rise in interest in recent decades. On the one hand, this was due to the rejuvenation of canonical quantum gravity in the form of *loop quantum gravity* from the late 1980s onwards.[1] At about the same time, the covariant *gravitational path integral* was given a new, non-perturbative lease of life in terms of "dynamical triangulations". Motivated originally by the search for a non-perturbative dynamics of curved, two-dimensional worldsheets in (bosonic) string theory, this dynamical lattice formulation provides a powerful computational tool for evaluating gravitational path integrals quantitatively: analytically in two, and numerically in higher dimensions. – The focus of the present article will be on this latter development, arguably the conceptually most straightforward and methodologically minimalist extension of the standard perturbative and covariant quantum field-theoretic formulation of gravity. We will explain how it may lead to the construction of a viable theory of quantum gravity, valid on all scales, without

[1] Curiously, this ansatz also postulates the fundamental character of certain one-dimensional "closed-string" (a.k.a. "loop") excitations in the quantum theory.

running into contradictions vis-à-vis the perturbative non-renormalizability of the theory.

In the late 1970s, Weinberg outlined a scenario, coined *asymptotic safety* [1], for how quantum field theories which are not power-counting renormalizable around a trivial Gaussian fixed point could under certain, general conditions still make sense, just like ordinary renormalizable theories. In particular, an asymptotically safe theory is characterized by only a *finite* number of coupling constants, whose values will be determined by comparison with experiment or observation. The asymptotic freedom scenario is naturally described in the language of quantum field theory and the renormalization group. It is characterized by the presence of an ultraviolet fixed point in the infinite-dimensional coupling constant space of a theory, with the property that in the fixed point's neighbourhood the dimension of the subspace of attraction is infinite-dimensional, with finite co-dimension. This co-dimension coincides with the number of free parameters of the theory that need to be fixed by experiment. Such a UV fixed point therefore attains a similar status to that of the Gaussian fixed point of a renormalizable theory. The snag is that the tools of the perturbative theory are usually not sufficient to find such ultraviolet fixed points – if they exist for a given theory – and to study their neighbourhoods.

To illustrate the implications of the presence of such a fixed point (in a somewhat simplistic fashion), let us introduce the dimensionless coupling

$$\tilde{G}(E) := GE^2. \tag{1}$$

A fixed point in this context always refers to the behaviour under a change of scale E of a dimensionless, energy-dependent function like $\tilde{G}(E)$. The dimensionful quantity G in (1) can at this stage still be thought of as a (classical, low-energy) coupling *constant* of mass dimension -2. Let the behaviour of $\tilde{G}(E)$ be dictated by a beta function $\beta(\tilde{G})$ according to

$$E\frac{\mathrm{d}\tilde{G}}{\mathrm{d}E} = \beta(\tilde{G}), \quad \text{with} \;\; \beta(\tilde{G}) = 2\tilde{G} - 2\omega\tilde{G}^2, \tag{2}$$

for some real parameter ω. It is immediately clear that for $\omega \neq 0$, $G = const$ is no longer a solution to (2). For consistency, G has to acquire a non-trivial E-dependence and therefore becomes a *function* $G(E) = \tilde{G}(E)/E^2$. In (2) we have chosen the simplest non-trivial beta function such that (i) in the limit of low energy, $E \to 0$, $G(E)$ goes to a constant (which we will continue to call G), and (ii) for $E \to \infty$, $\tilde{G}(E)$ goes to a non-trivial UV fixed point. Explicitly, the solution to the differential equation in (2) can be stated as

$$G(E) = \frac{G}{1 + \omega G E^2}, \tag{3}$$

from which we can read off the location of the UV fixed point at $\tilde{G} = 1/\omega$, the non-trivial zero of the beta function. An important feature of this solution is that the coupling constant $G(E)$ goes to zero at the UV fixed point.

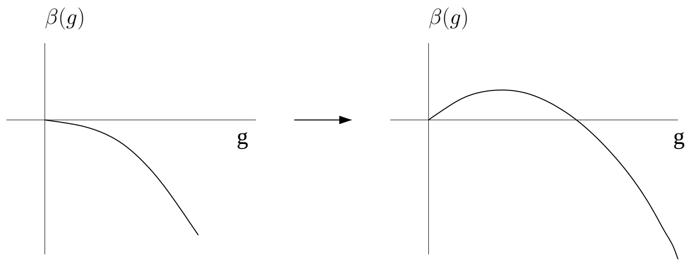

Figure 1: Changing an asymptotically free theory to an asymptotically safe one by increasing its dimension from d to $d+\varepsilon$ results in a shift of its ultraviolet fixed point to a value $g > 0$.

In case the above example should appear somewhat ad hoc, it can be understood as arising from a more general construction, which starts from an asymptotically *free* theory in d dimensions. Fig. 1 (left) illustrates the corresponding (negative) beta function of the coupling g, together with a Gaussian UV fixed point at $g = 0$. If this theory is "lifted" to $d+\varepsilon$ dimensions – assuming that such a perturbation in the dimension is well defined, at least for small $\varepsilon > 0$ – its beta function will change according to

$$\beta(g) \to \rho(\varepsilon)g + \beta(g), \tag{4}$$

where $\rho(\varepsilon)$ is the (positive) amount by which the mass dimension of the coupling g *de*creases as a result of the dimensional increase by ε. (Our previous example, whose beta function was defined in relation (2), corresponds to $\rho = 2$.) Note that the Gaussian UV fixed point of the original theory has become a non-trivial UV fixed point away from zero in the higher-dimensional theory, while $g = 0$ has been turned into an infrared fixed point, as illustrated by Fig. 1.

The theories we have discussed so far – four-Fermi theory, non-linear sigma model and Einstein gravity – display a similar behaviour in the sense that they are asymptotically free, renormalizable theories in spacetime dimension $d = 2$. Trying to make sense of them beyond dimension 2 by way of a $2+\varepsilon$-expansion, one encounters the situation depicted in Fig. 1. Of course, one may formally set $\varepsilon = 2$ in such an expansion, as would be needed to reach the dimension $d = 4$ of physical spacetime, but the validity of the perturbative expansion for such large values of ε would need to be established to take the results seriously, and a priori appears perhaps rather doubtful.

Non-trivial UV-complete extensions to $d = 4$ of the four-Fermi interaction or the non-linear sigma model are not known and presumably do not exist. As mentioned above, we should rather think of them as effective theories, which happen to describe certain low-energy properties of more fundamental theories with more and different fundamental excitations. Still, it is difficult to draw any conclusions from this for general relativity, the theory we are interested in, which is after all very different physically: exactly the degrees of freedom that are fixed in all other theories, those of spacetime itself, become dynamical in gravity. Much work has gone into trying

to show that four-dimensional gravity possesses an ultraviolet fixed point with the requisite properties, either in terms of the $2+\varepsilon$-expansion [2] or by using general renormalization group techniques [3].

In what follows, we will not be concerned with the details of these efforts, but with the question of how the hypothesis of asymptotically safe gravity may be tested independently and non-perturbatively by using standard field-theoretic tools and by formulating quantum gravity via a lattice regularization.

2 A lattice theory for gravity

A number of issues have to be addressed when representing gravity on a lattice. Is it possible in principle to construct a well-defined lattice regularization of gravity with a UV lattice cut-off, which can be removed in a controlled way to obtain a continuum limit (whatever this may turn out to be)? The answer is *yes*. More precisely, the issue is not so much how to represent gravity on a lattice, but how to represent a theory as a lattice theory whose standard continuum formulation in terms of local fields is diffeomorphism-invariant, a vast gauge invariance closely related to the differentiable structure of the underlying manifold and its description in terms of local coordinate charts.

For the geometric degrees of freedom of the gravitational theory this can be done by viewing the lattice itself as representing directly a (piecewise linear) geometry. The key point is that such a geometry can be described uniquely without ever introducing coordinates, thus circumventing the associated redundancy of having to choose any particular set of coordinates. A convenient choice is to use lattices which are triangulations, in the sense of consisting of d-simplices, triangular building blocks which are d-dimensional generalizations of flat triangles (=2-simplices). Assuming the interior of a d-simplex to be flat, its geometry is uniquely specified by giving the lengths of its $d(d+1)/2$ one-dimensional edges or links. Together with the information of how the simplices are "glued together" (that is, how $(d-1)$-dimensional boundary simplices are identified pairwise) to form a triangulated manifold, this suffices to compute all geometric information, including distances, geodesics, volumes etc. without using coordinates. Important for our path integral representation, Regge observed that the curvature of such a piecewise linear geometry is in a natural way located on its $(d-2)$-dimensional subsimplices (the "hinges"). By the same token, the scalar curvature term of the Einstein action of such a geometry is given by the sum over all hinges of the deficit angle around each hinge, multiplied by the hinge's volume [4].

In our construction of a theory of quantum gravity, the lattice-regularized path integral over geometries thus becomes the sum over such triangulations, with weight depending on the Regge implementation of the Einstein action. Precisely which class of triangulations should we sum over in the path integral? When applying Regge calculus to classical gravity one uses a fixed lattice, in the sense of leaving

the connectivity of its constituent simplicial building blocks unaltered. This still allows the curvature of the triangulation to be changed – for example, to optimally approximate that of a given smooth geometry – by changing the lengths of its one-dimensional edges.

When using the piecewise linear geometries in a path integral, the task is different. Firstly, we do not expect the individual path integral configurations to be smooth, but only continuous, in the same way as the paths in the path integral of a quantum-mechanical particle are continuous but in general non-smooth[2]. Similarly, the piecewise linear geometries are a subset of all continuous spacetime geometries. Note that we can even restrict ourselves to a subset of piecewise linear geometries as long as it is suitably dense in the set of all geometries. More precisely, when the lattice spacing goes to zero, we require the expectation values of observables, again suitably defined on the piecewise linear geometries, to converge to the value they would take in the continuum quantum field theory (which we assume exists). In contrast with the aim of the classical theory, we are therefore not trying to approximate any *particular* geometry by our lattice geometries, but to span the whole set of geometries.

In this context a specific subset of piecewise linear geometries has proved to be very useful, namely, the triangulations whose edges have all the same length, a, say. One can characterize this set of geometries as being constructed from gluing together *equilateral* simplicial building blocks in all possible ways, compatible with certain constraints (typically, a fixed topology and fixed boundary components). Consequently, the variation in geometry (the way in which the geometric degrees of freedom are encoded) is linked to the mutual connectivity of the building blocks created by the gluing and not to variations in the link lengths, giving rise to the name *Dynamical Triangulations* (DT) [5, 6, 7]. From a path-integral perspective this approach has the advantage that distinct triangulations correspond to physically distinct geometries. Summing over this DT ensemble of geometries may therefore lead directly to the correct continuum measure in the limit that the UV cut-off is taken to zero, $a \to 0$. By contrast, treating the triangulations classically à la Regge, with fixed lattice connectivity and variable link lengths, still contains redundancies, in the sense that many different lattice configurations can correspond to the same physical geometry (see [8] and references therein). For illustration, consider a rectangle in the two-dimensional plane and triangulate its interior. Clearly, the interior vertices can be moved around locally in the plane without changing the flat geometry of the rectangle. However, since all of these are different as Regge triangulations, this leads to a severe overcounting in the path integral of quantum Regge calculus, for which there is currently no known fix.

Most importantly, the viability of the DT lattice regularization has already been demonstrated in a non-trivial case, that of gravity (coupled to matter) in two dimensions. As mentioned above, two-dimensional gravity is a renormalizable quantum

[2]in fact, with unit probability they are nowhere differentiable

field theory and various observables can be calculated analytically [9]. The dynamically triangulated two-dimensional lattice theory can also be solved, a number of observables can be calculated analytically and its continuum limit, taking the lattice spacing $a \to 0$, can be taken [10]. Remarkably, results from the two different calculations can be compared and are found to agree. We conclude that *it is possible to provide a viable lattice regularization of a diffeomorphism-invariant quantum theory of geometries.*

One may object that this two-dimensional theory has little to do with true gravity in four spacetime dimensions; to start with, it has no propagating gravitons. However, we would like to argue that it is much more a theory of fluctuating geometries than one would ever expect of the four-dimensional theory. Because there is no Einstein-Hilbert action in two dimensions (it is topological), each configuration contributes in the path integral with the same weight, which is a maximally quantum situation. This is borne out by the analytic solutions of this model, which show the two-dimensional geometries as wildly quantum-fluctuating. Nevertheless the lattice theory has no problem in reproducing the correct diffeomorphism-invariant continuum theory, also known as quantum Liouville gravity.

2.1 Observables

How to define what does and does not constitute an "observable" in quantum gravity, and how to construct and evaluate observables in any given formulation are physical questions of central importance . What we would like to highlight here is that a beautiful aspect of a geometric lattice formulation of quantum gravity of the type we are considering is that it forces one to address such questions head-on. It is not possible to hide behind some "expansion around flat spacetime", but one is forced to think in terms of physical "rods and clocks", much in the spirit of Einstein's classical theory.

Let us discuss the basic objects of any quantum field theory, namely, the correlators of local quantum operators $\mathcal{O}(x)$. Such correlators are important ingredients in constructing S-matrix elements, i.e. observables in quantum field theory on a fixed background. Also in conventional lattice theories, correlators play a crucial role in showing that a lattice theory has a continuum limit when the lattice spacing goes to zero.

Consider some lattice scalar field theory, and let $\mathcal{O}(x_n)$ be an operator at lattice spacetime coordinate $x_n = n \cdot a$, where a is the lattice spacing and n the integer-valued lattice coordinate. In general, we expect the correlator to fall off exponentially,

$$-\log\langle\mathcal{O}(x_n)\mathcal{O}(x_m)\rangle \sim |n-m|/\xi(g_0) + o(|n-m|), \tag{5}$$

where g_0 is the bare lattice coupling and $\xi(g_0)$ the correlation length in lattice spacings. The standard procedure for a lattice system is to take the continuum limit

at a second-order phase transition point g_0^c, where the correlation length diverges like

$$\xi(g_0) \propto \frac{1}{|g_0 - g_0^c|^\nu}, \qquad a(g_0) \propto |g_0 - g_0^c|^\nu. \tag{6}$$

Eq. (6) tells us at what rate we should scale the lattice spacing to zero in the limit $g_0 \to g_0^c$, in order to find an exponential decay in the continuum, when the lattice correlation diverges, but the (dimensional) physical length $x_n - x_m = (n-m)a$ is kept constant,

$$m_{ph} a(g_0) = 1/\xi(g_0), \qquad \mathrm{e}^{-|n-m|/\xi(g_0)} = \mathrm{e}^{-m_{ph}|x_n - x_m|}. \tag{7}$$

Eq. (7) illustrates the fact that dimensionful observables, like the physical mass m_{ph}, are defined by the *approach* to the critical point, not *at* the critical point.

The existence of a critical point and an associated divergent correlation length constitute the backbone of the Wilsonian renormalization group approach to quantum field theory. Since we are appealing to this Wilsonian approach by asking whether asymptotic safety is realized, it is important to understand whether it can be applied to quantum gravity at all. A first step in this direction is to understand whether suitable correlators and a correlation length can be defined in a diffeomorphism-invariant theory like quantum gravity. To start with, how can we define the distance between two points in a path integral where we integrate over the geometries defining this distance?

In flat d-dimensional spacetime, let us rewrite the correlator of a scalar field $\phi(x)$, say, in the form

$$\langle \phi\phi(R)\rangle_V \equiv \frac{1}{V}\,\frac{1}{s(R)} \int \mathcal{D}\phi\, e^{-S[\phi]} \int d^d x \int d^d y\; \phi(x)\phi(y)\; \delta(R - |x-y|). \tag{8}$$

As indicated, this expression depends on a chosen distance R, but no longer on specific points x and y, which instead are integrated over. The integrand can be read "from right to left" as first averaging over all points y at a distance R from some fixed point x, normalized by the volume $s(R)$ of the spherical shell of radius R, and then averaging over all points x, normalized by the total volume V of spacetime. We assume translational and rotational invariance of the theory and that V is so large that we can ignore any boundary effects related to a finite volume.

This definition of a correlator is of course non-local, but unlike the underlying locally defined correlator has a straightforward diffeomorphism-invariant generalization to the case where gravity is dynamical, namely,

$$\begin{aligned}\langle \phi\phi(R)\rangle_V \;\; &\equiv \;\; \frac{1}{V}\int \mathcal{D}[g] \int \mathcal{D}_{[g]}\phi\; \mathrm{e}^{-S[g,\phi]}\; \delta\Big(V - \int d^d x \sqrt{\det g}\Big)\cdot \\ &\int d^d x \int d^d y\; \frac{\sqrt{\det g(x)}\,\sqrt{\det g(y)}}{s_{[g]}(y,R)}\; \phi(x)\phi(y)\; \delta(R - D_{[g]}(x,y)),\end{aligned} \tag{9}$$

which now includes a functional integration over geometries[3] $[g]$, and dependences of the action, measures, distances and volumes on $[g]$. Can the definition (9) be implemented meaningfully to define correlators in a quantum gravity theory? The answer is yes, and a two-dimensional example can again be used to demonstrate this. Namely, there are analytic predictions for the behaviour of the propagators of certain matter theories coupled to two-dimensional Euclidean gravity [9], which have been shown to be reproduced by numerical simulations of the corresponding lattice theory [11]. By the way, their behaviour is quite different from that of the flat space correlators, another manifestation of the fact that two-dimensional gravity is a theory of strong geometric fluctuations.

2.2 Time-slicing and baby universes

An interesting aspect that can be analyzed in detail in the solvable two-dimensional quantum theory of fluctuating geometry is that of proper time . One usually considers a situation where the rotation to Euclidean signature has taken place and "proper time" is simply given by "geodesic distance". In this setting, a closed one-dimensional spatial universe of fixed "time" is simply a loop of length ℓ. In the corresponding quantum theory one can ask for the amplitude for a universe of length ℓ_1 to "propagate" to another one of length ℓ_2 in proper time t. More precisely, the outgoing loop of length ℓ_2 is said to have a proper-time (in this case a geodesic) distance t to the incoming loop of length ℓ_1 if each point on ℓ_2 has geodesic distance t to ℓ_1. (The geodesic distance from a point to a set of points is defined as the minimum of the geodesic distances from the point to the points in the set.)

Fig. 2 shows a typical geometry in the path integral contributing to the corresponding amplitude $G(\ell_1, \ell_2; t)$. It will often be convenient to work with its Laplace transform,

$$G(x, y; t) = \int_0^\infty \int_0^\infty d\ell_1 d\ell_2 \, \mathrm{e}^{-x\ell_1 - y\ell_2} \, G(\ell_1, \ell_2; t). \tag{10}$$

We can view x and y in this expression as *boundary cosmological constants*, since $x \cdot \ell$ would be the action of a one-dimensional "spacetime" of volume ℓ and cosmological constant x.

As shown in [12], the amplitude $G(x, y; t)$ satisfies the remarkably simple equation

$$\frac{\partial G(x, y, t)}{\partial t} = \frac{\partial (W(x) G(x, y, t))}{\partial x}, \tag{11}$$

where $W(x)$ is the Hartle-Hawking disk amplitude, which in two-dimensional Euclidean gravity is given by [10]

$$W(x) = (x - \frac{1}{2}) \sqrt{x + \sqrt{\Lambda}}. \tag{12}$$

[3]in accordance with standard notation, $[g]$ denotes an equivalence class of metrics g under the action of the diffeomorphism group

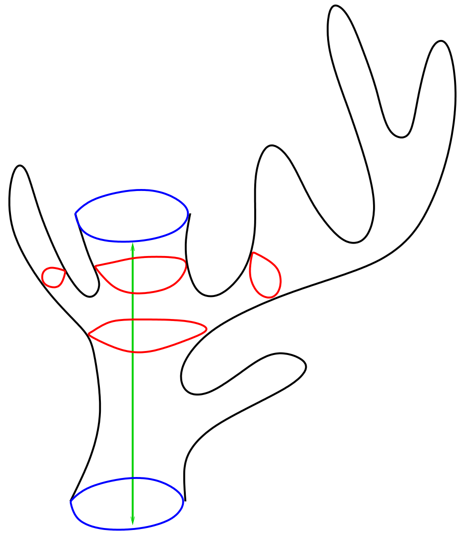

Figure 2: Incoming and outgoing boundary loops of length ℓ_1 and ℓ_2, separated by a geodesic distance t, and a typical interpolating geometry of cylinder topology which contributes to the amplitude $G(\ell_1, \ell_2; t)$ in Euclidean signature. The additional loops drawn onto the interior geometry consist of points which share the same distance to the incoming loop. As indicated by the upper set of three loops, there can be many disconnected loops at a given distance to the incoming loop.

As is clear from Fig. 2, space can branch out into many disconnected parts (i.e. change its topology) as a function of proper time t, giving rise to "baby universes". The appearance of baby universes on all scales leads to the two-dimensional quantum spacetime being fractal , with Hausdorff dimension $d_h = 4$ [12, 13].

Rather amazingly, it is possible to integrate analytically over these baby universes, resulting (for each time history) in a spacetime with a proper-time foliation and no baby universes [14]. Alternatively, the expression for the loop-loop propagator without baby universes can be obtained directly by summing over a class of two-dimensional spacetimes which from the outset lack baby universes, provided one redefines the coupling constants suitably [15]. This latter procedure can be implemented also at the regularized level in terms of a set of "*causal* dynamical triangulations" (CDT), to be distinguished from the larger class of merely "dynamical triangulations" (DT), which served as carrier space for the Euclidean gravitational path integral [15].

The resulting theory has a well-defined Hamiltonian and corresponding unitary proper-time evolution. The explicit map between the cosmological constants of DT and CDT turns out to be non-analytic,

$$\tilde{\Lambda}_{cdt} = \sqrt{\Lambda_{dt}}, \qquad \tilde{x}_{cdt} = \sqrt{x + \sqrt{\Lambda_{dt}}}, \tag{13}$$

where we have denoted the CDT-analogues of the couplings with a subscript and tilde. Consequently, in CDT both lengths and areas acquire a dimensionality dif-

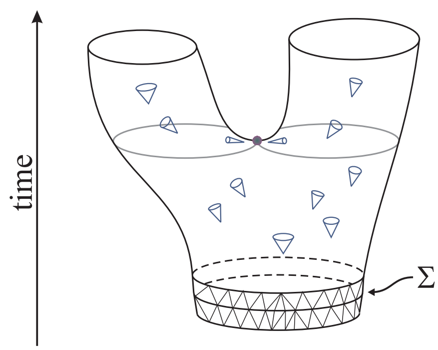

Figure 3: The light cone structure (and therefore the underlying Lorentzian geometry) becomes degenerate in points where space splits in two.

ferent from that found in the DT ensemble of spacetimes and in Liouville gravity. When using the CDT ensemble, also the Hausdorff dimension changes from 4 to 2, the canonical value for ordinary smooth two-dimensional spacetimes.[4]

The CDT loop-loop propagator satisfies the equation

$$\frac{\partial \tilde{G}(\tilde{x},\tilde{y},t)}{\partial t} = \frac{\partial((\tilde{x}^2 - \tilde{\Lambda}_{cdt})\tilde{G}(\tilde{x},\tilde{y},t))}{\partial \tilde{x}}, \tag{14}$$

and the Hamiltonian governing the (proper-) time evolution is given by

$$\tilde{G}(\tilde{\ell}_1,\tilde{\ell}_2,t) = \langle \tilde{\ell}_2 | \mathrm{e}^{-t\hat{H}} | \tilde{\ell}_1 \rangle, \qquad \hat{H} = -\tilde{\ell}\frac{d^2}{d\tilde{\ell}^2} + \tilde{\Lambda}_{cdt}\tilde{\ell}, \tag{15}$$

while the CDT Hartle-Hawking wave function (which is derived from the propagator $\tilde{G}$ [15]) satisfies

$$\hat{H}\tilde{W}_{cdt}(\tilde{\ell}) = 0. \qquad \text{(Wheeler-DeWitt)} \tag{16}$$

Above, our first way of deriving this formulation was as a kind of "effective" theory: we started from the set of all Euclidean two-dimensional geometries of a fixed topology. These geometries are "isotropic" in the sense that they do no carry any a priori preferred direction. We then superimposed a notion of proper time on them and integrated out part of the degrees of freedom. However, when starting in the physically correct *Lorentzian* signature, one can formulate a general principle which excludes geometries whose *spatial* topology is not constant in time [16]. The point is that spatial topology changes are associated with causality violations of one kind or other. This is illustrated by the "trouser geometry" depicted in Fig. 3. As is clear from the embedding of this two-dimensional spacetime in flat Minkowski space, with time pointing upward, there must be at least one point near the crotch of the trousers where the tangent plane is exactly horizontal and the light cone therefore

[4]A word of warning: the coincidence in Hausdorff dimension does *not* allow one to conclude that the quantum geometry of two-dimensional CDT in any way approximates a smooth classical manifold; in fact, it does not.

degenerate. Note that imposing causality conditions on the geometry to eliminate such configurations only makes sense in the presence of a Lorentzian metric and cannot even be formulated in a purely Euclidean theory, in the absence of any extra structure.

By the same token, one can take as domain of the path integral the set of all Lorentzian piecewise flat triangulations whose causal structure is well defined, and where in particular no changes of spatial topology are allowed to occur. The set of *causal dynamical triangulations* (CDT) – which can be defined in any dimension (not just $d = 2$) – obeys a strong version of causality of this kind, which is implemented by requiring each triangulation to be the product of a one-dimensional "triangulation" (a line with equidistant points), representing discrete proper time, and other triangulated degrees of freedom, representing the spatial directions of the geometry, which may be thought of as triangulated fibres over a one-dimensional base space.[5] As an added bonus, each triangulation in the class of CDT can be analytically continued to Euclidean signature, and the associated gravitational Regge actions satisfy the standard relation between actions defined in spacetimes of Lorentzian and Euclidean signatures, namely,

$$iS_{Lorentzian} \mapsto -S_{Euclidean}. \tag{17}$$

Despite the fact that the actions obey (17), the Lorentzian theory defined on CDT geometries will even after this "Wick rotation" be *distinct* from the full Euclidean theory, because not every Euclidean triangulation is the image of a causal, Lorentzian one. The subclass of Euclidean geometries that are in the image can be obtained "surgically" as explained above, by superimposing a notion of proper time on each Euclidean triangulation and then removing all of its baby universes associated with spatial topology changes. The two-dimensional case is sufficiently simple to allow us to perform the calculation in either way, by starting from a path integral over all Euclidean geometries and removing baby universes, or by starting from a path integral over causal (CDT) geometries and rotating it to Euclidean signature. Both results agree after a redefinition of the coupling constants. Let us note in passing that our formulation – not only in dimension 2, but also in higher dimensions – has a couple of characteristics reminiscent of so-called Hořava-Lifshitz gravity , namely, the use of a preferred time foliation and a unitary time evolution. We will return to this subject in Sec. 4 below.

2.3 CDT in higher dimensions

It is not known whether the above-described procedure of integrating out baby universes in $d = 2$ can be generalized to higher dimensions in a simple and useful way. It implies that at this stage we have two a priori unrelated lattice gravity theories in dimension $d > 2$, one purely Euclidean based on DT and one Lorentzian

[5]Product triangulations, of which this is a particular instance, were investigated in [17], see also [18].

based on CDT. The latter starts out in physical, Lorentzian signature, and imposes local causality conditions (nondegeneracy of local light cones) and a proper-time time foliation.[6] For calculational purposes, these lattice configurations are then rotated to Euclidean signature and the path integral over this class can in principle be performed. Of course, since the physics one hopes to describe ultimately by these theories has Lorentzian character, one will have to perform an "inverse Wick rotation" back to Lorentzian spacetime eventually, never mind whether the computation at an intermediate step took place in a purely Euclidean or in a Euclideanized Lorentzian framework.

The simplest implementation of Euclidean DT based on the lattice Regge version of the Einstein-Hilbert action (the inclusion of a cosmological term being understood) does not seem to lead to a theory with an interesting continuum limit. Even if this is the case, it is in principle possible that by adding more terms to the bare lattice action and suitably tuning the associated new coupling constants, an interesting continuum theory may emerge after all. This possibility has been investigated in the past [19], as well as more recently [20], but there is no conclusive evidence at this point that these modified Euclidean models can reproduce the physical properties of quantum gravity from CDT, the Lorentzian lattice gravity theory to which we will turn next (see also [21] for a variety of reviews of the subject).

Fig. 4 illustrates the general construction of a four-dimensional CDT triangulation. We take space to be compact and with the simplest topology, that of the three-sphere S^3. In addition, we assume a discrete proper-time foliation and represent the spatial geometry at each integer proper time t by a three-dimensional simplicial manifold, given as some configuration of Euclidean DT in terms of equilateral tetrahedra. By assumption, the tetrahedra are flat in the interior, which means that their geometric properties are uniquely specified by the length of their edges, which is some number $a_s > 0$ (the same for all edges). To obtain a four-dimensional Lorentzian simplicial manifold with signature $(- + ++)$, we still must fill in all intervals $[t, t+1]$ between consecutive spatial slices. This can be done by using two types of geometrically distinct four-simplices, which again by assumption are flat in the interior, but this time with Lorentzian signature. The two different types are the (4,1)- and the (3,2)-simplex depicted in Fig. 4, together with their time-reversed counterparts. The (4,1)-simplex has as its "base" one of the spatial tetrahedra contained in the triangulated constant-time slice. (The "4" in the label (4,1) refers to the four vertices contained in slice t that span this tetrahedron; similarly, the "1" refers to the single vertex shared with slice $t+1$. An analogous labeling has been used for the (3,2)-simplex.) All that remains to be done to fix the geometry of the four-simplices is to assign lengths to the edges that have their end points in adjacent slices, and whose time labels therefore differ by one unit. We choose them to be all time-like and of equal (absolute) length $a_t > 0$, which in our

[6] Note that there is no strict physical requirement that individual path integral histories *must* be causal; individual histories are not physical, observable quantities, only expectation values computed in the ensemble of histories are.

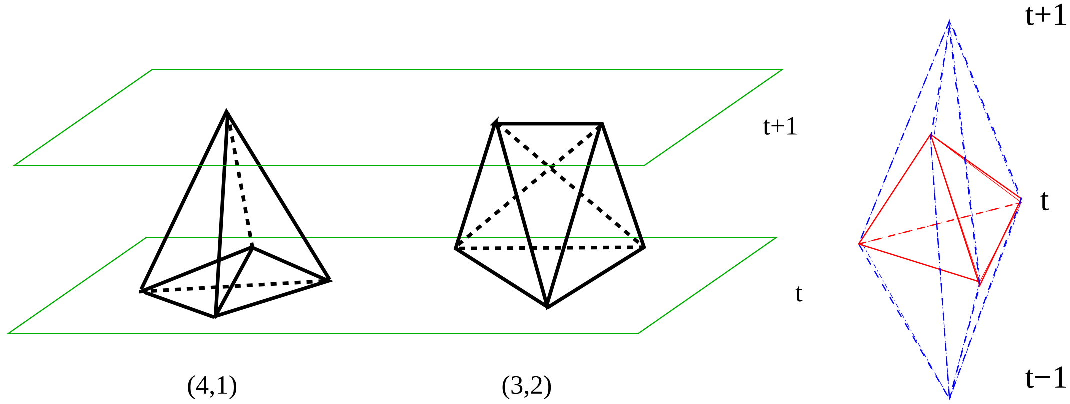

Figure 4: A triangulation in CDT consists of four-dimensional triangulated layers assembled from (4,1)- and (3,2)-simplices, interpolating between adjacent integer constant-time slices (left), which in turn are triangulations of S^3 in terms of equilateral tetrahedra. Each purely spatial tetrahedron at time t forms the interface between two (4,1)-simplices, one in the interval $[t-1,1]$, and the other in $[t,t+1]$, as illustrated on the right. Although a (3,2)-simplex shares none of the five tetrahedra on its surface with a constant-time slice (the tetrahedra are all Lorentzian), it is nevertheless needed in addition to the (4,1)-building block to obtain simplicial manifolds with a well-defined causal structure.

signature convention implies that their squared edge length is given by $-a_t^2$.[7]

Our choice of causal geometries and length assignments has the added benefit that we can define a map that uniquely maps each Lorentzian CDT history to a Euclidean DT history. Let us start by parametrizing the relative length of the two lattice parameters a_s and a_t by a positive real number α defined by $\alpha := -a_t^2/a_s^2$. Performing a rotation $\alpha \to -\alpha$ in the complex lower-half plane can be interpreted as changing all time-like length assignments of lattice links to space-like ones according to

$$a_t^2 = -\alpha a_s^2 \quad \to \quad a_t^2 = \alpha a_s^2. \tag{18}$$

In order that the Euclidean four-simplices obtained after this rotation satisfy triangle inequalities we require $\alpha > 7/12$. The resulting triangulation represents a piecewise linear manifold with *Euclidean* signature. If one writes the Lorentzian Regge action as a function of a single lattice parameter $a := a_s$ and of α, the action behaves under the rotation (18) as one would expect naïvely from a rotation from Lorentzian to Euclidean spacetime, namely,

$$iS_L[\alpha] = -S_E[-\alpha]. \tag{19}$$

The prescription (18) leading to (19) is the "Wick rotation" we had in mind in our

[7]Note that a_t gives us an approximate distance measure between adjacent spatial slices labelled by integer-t, where the distance of a point in slice $t+1$ to slice t is defined as the length of the longest geodesic from the point to the slice.

earlier discussion in Sec. 2.2. It transforms the original Lorentzian path integral with complex weights $\mathrm{e}^{iS_L(T)}$ to one with real weights $\mathrm{e}^{-S_E(T)}$, where by slight abuse of notation we use the same symbol T to denote the initial triangulation (with Lorentzian edge length assignments) and the one after rotation (which has identical connectivity, but purely Euclidean edge length assignments). Modulo the sign flip for the length assignments, the domain of the Euclideanized path integral is the same set $\mathcal{T} = \{T\}$ of triangulations as that of the original Lorentzian path integral. The set $\mathcal{T}$ is of course smaller than the set of *all* Euclidean triangulations one would obtain by gluing together the same Euclideanized building blocks, because it still carries an imprint of the causality conditions imposed on the Lorentzian triangulations.

The fact that in DT and CDT we use *standardized* building blocks to construct the triangulations means that the Regge action takes on a very simple functional form. For the special case $|\alpha| = 1$ we have after the Wick rotation only a single type of building block, the equilateral four-simplex with all link lengths equal to $a \equiv a_s$. The Regge form of the Einstein-Hilbert action becomes

$$S_E[-\alpha = -1; T] = -\kappa_0 N_0(T) + \kappa_4 N_4(T), \tag{20}$$

as is well-known from Euclidean DT quantum gravity. In (20), $N_0(T)$ denotes the number of vertices in the triangulation T, and $N_4(T)$ the number of its four-simplices. The coupling κ_0 is related to the gravitational coupling constant G via $1/\kappa_0 \propto Ga^2$, and κ_4 should be identified with $a^4\Lambda/G$, where Λ is the cosmological constant.

Whenever $|\alpha| \neq 1$, we retain the two different building blocks (of type (4,1) and (3,2)) after the rotation, and the action will depend on their total numbers, $N_4^{(4,1)}$ and $N_4^{(3,2)}$, separately instead of only on their sum $N_4 = N_4^{(4,1)} + N_4^{(3,2)}$. It is convenient to parametrize the resulting Euclideanized Regge action in the form

$$\begin{aligned} S_E[-\alpha; T] \;&=\; -(\kappa_0 + 6\Delta)N_0(T) + \kappa_4 \Big(N_4^{(3,2)}(T) + N_4^{(4,1)}(T)\Big) + \\ &\quad\; \Delta\Big(N_4^{(3,2)}(T) + 2N_4^{(4,1)}(T)\Big), \end{aligned} \tag{21}$$

where the asymmetry parameter Δ is a function of α such that $\Delta(\alpha{=}1) = 0$.

We note that Δ appears in (21) on a par with the other two coupling constants, κ_0 and κ_4. In what follows, we will treat it as a third independent coupling constant. The reason for doing this – despite the fact that it has no immediate interpretation in the Einstein-Hilbert action – is that in the region of phase space (the space spanned by the three couplings κ_0, κ_4 and Δ) where we observe interesting, apparently continuum physics, the entropy of geometries is as important as the contributions coming from the bare action term. To make this more explicit, one can rewrite the Euclidean partition function of the theory as a sum over the counting variables

$N_4^{(4,1)}$, $N_4^{(3,2)}$ and N_0 according to

$$\begin{aligned} Z(\kappa_0, \kappa_4, \Delta) &= \sum_T \mathrm{e}^{-S_E[T]} \qquad (22) \\ &= \sum_{N_4^{(4,1)}, N_4^{(3,2)}, N_0} \mathrm{e}^{-S_E[N_4^{(4,1)}, N_4^{(3,2)}, N_0]} \mathcal{N}(N_4^{(4,1)}, N_4^{(3,2)}, N_0), \end{aligned}$$

where $\mathcal{N}(N_4^{(4,1)}, N_4^{(3,2)}, N_0)$ is the number of triangulations with $N_4^{(4,1)}$ four-simplices of type (4,1), $N_4^{(3,2)}$ four-simplices of type (3,2) and N_0 vertices. Introducing the notation $c_1 = N_0/N_4^{(4,1)}$ and $c_2 = N_4^{(3,2)}/N_4^{(4,1)}$, the leading-order behaviour of this combinatorial quantity in the large-volume limit is known to be of the form

$$\mathcal{N}(N_4^{(4,1)}, N_4^{(3,2)}, N_0) = \mathrm{e}^{f(c_1,c_2)N_4^{(4,1)} + \text{s.l.}}, \qquad (23)$$

where "s.l." denotes subleading terms in $N_4^{(4,1)}$, and the c_i typically have some boundedness properties. Since in the same limit the action (21) can be similarly approximated by $S_E = \tilde{f}(c_1, c_2)N_4^{(4,1)} + \text{s.l.}$, it implies that in the region of phase space where the four-volume can become large, both $\mathcal{N}$ and e^{-S_E} have the same functional form and are potentially of the same magnitude. It turns out that this is the same region where we observe interesting continuum-like physics. Because of contributions from both "energy" and "entropy", it is clear therefore that the effective action governing physics in this non-perturbative region can be very different from the "naïve" Einstein-Hilbert action, justifying our inclusion of Δ as a tunable parameter in the bare action.

To summarize: taking as our starting point spacetimes with Lorentzian signature, we can consider the transition amplitude between an initial and a final spatial three-geometry, $[g_{\mathrm{i}}^{(3)}]$ and $[g_{\mathrm{f}}^{(3)}]$ separated by a proper time t. We can then regularize the theory, using CDT, representing three-geometries by equilateral Euclidean triangulations and spacetime geometries by causal, Lorentzian triangulations with a discrete proper-time foliation. In the CDT framework, each of the latter can be rotated to Euclidean signature, leading to a regularized, Euclideanized sum-over-histories. What remains to be done is to "remove the regulator", that is, take the lattice spacing a to zero. Denoting the initial and final spatial triangulations by $T_i^{(3)}$ and $T_f^{(3)}$, we thus arrive at the prescription

$$G_E([g_{\mathrm{i}}^{(3)}], [g_{\mathrm{i}}^{(3)}], t, \kappa_0, \kappa_4, \Delta) := \lim_{a\to 0} \sum_{T: T_i^{(3)} \to T_f^{(3)}} \mathrm{e}^{-S_E[T]}, \qquad (24)$$

which can be viewed as the four-dimensional generalization of the two-dimensional loop-loop amplitude $\tilde{G}(\tilde{\ell}_1, \tilde{\ell}_2, t)$ introduced in (15) above. For a more detailed description of the CDT construction we refer the interested reader to [22, 23, 24].

3 The phase diagram

Contrary to the situation in two dimensions, we cannot calculate the amplitude (24) analytically. However, we can extract a lot of non-trivial, non-perturbative information by performing Monte Carlo computer simulations . This will usually start with an investigation of the structure of the space of coupling constants (the "phase space" of the underlying statistical system), in particular, trying to identify regions associated with a second-order phase transition, where according to standard lore one can hope to obtain continuum physics .

Let us highlight two technical aspects related to our implementation of the computer simulations. Firstly, rather than fixing specific boundary three-geometries $T^{(3)}$ at times 0 and t, we take time to be periodic. Although this is strictly speaking in contradiction with imposing causality (it introduces closed time-like curves), in practice it turns out to not affect results. The nature of the ground states of geometry is such that by choosing t sufficiently large – assumed from now on – the boundary condition becomes irrelevant.

Secondly, as we have discussed, the action (21) depends on three coupling constants, one of which, κ_4, can be identified with the cosmological coupling constant, multiplying the spacetime volume V in the action. In the computer simulations it is convenient to keep this four-volume fixed, which means that the cosmological constant does not really play a role. We compensate for this by performing separate simulations at different (fixed) spacetime volumes. From these we can in principle reconstruct results which depend on the cosmological constant via a Laplace transformation,

$$G(\kappa_4, ...) = \int_0^\infty dV \; \mathrm{e}^{\kappa_4 V} \, G(V, ...). \tag{25}$$

We are therefore left with two coupling constants, κ_0 and Δ. The corresponding phase diagram is shown in Fig. 5 [25] and exhibits three distinct phases, labelled A, B and C. Phase C appears to be the one relevant for continuum physics, because only there do we observe extended four-dimensional universes [26]. A careful numerical analysis reveals strong evidence that the transition between phases C and A is first order, whereas between phases C and B we find a second-order transition [27]. This very exciting result implies that the B-C phase transition line is a candidate for a region in the coupling-constant plane where genuine UV continuum limits may exist, defined by approaching specific points on the line. Conversely, moving away from the transition line into phase C corresponds to going towards an IR limit.

3.1 Phase C

The reason why phase C is related to extended four-dimensional spacetimes is illustrated in Fig. 6, which shows both a sample path-integral configuration generated by the computer during the Monte Carlo simulations, as well as the associated quantum observable, obtained by averaging in the ensemble. While of course we have

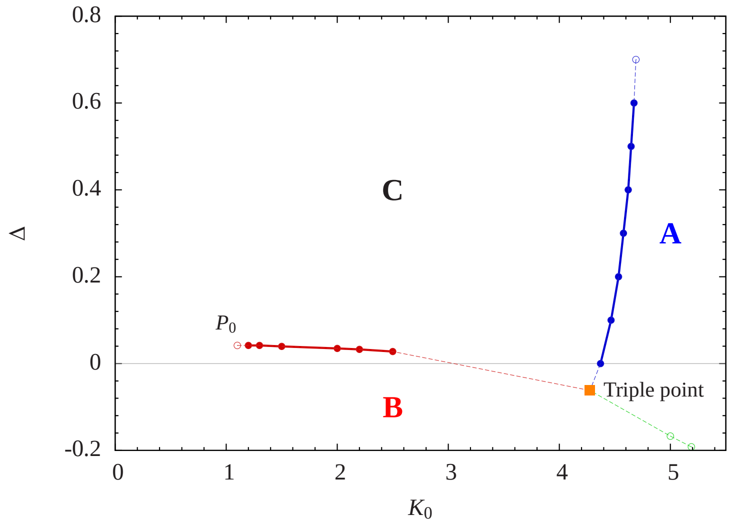

Figure 5: The phase diagram of CDT quantum gravity in the (κ_0, Δ)-plane.

access to the complete geometric information of the quantum spacetimes that are generated, only a single degree of freedom is depicted here, the three-volume of a spatial slice of the quantum spacetime as a function of proper time. The time extension in a given simulation is always fixed (in the case at hand to 80 discrete time steps). What we observe in Fig. 6 is that the universe does not make use of the full time interval available, but has a non-vanishing volume only on a connected subset of the time axis.[8]

A quantitative piece of evidence in favour of a *four*-dimensional extended universe is the fact that its time extension (not counting the stalk) scales like $N_4^{1/4}$ when the total discrete four-volume N_4 of the universe used in the simulations is varied. Similarly, its discrete three-volume $N_3(t)$ scales like $N_4^{3/4}$. Contrary to one's naïve expectations, these findings are highly non-trivial, because they have been derived in a non-perturbative, background-independent path integral formulation. The simplicial building blocks of our regularization *are* four-dimensional, but since assembling them is only dictated by the Boltzmann weight $e^{-S_E[T]}$ *without* any reference to a four-dimensional background, there is no reason why the resulting object, extrapolated to infinite lattice volume, should be four-dimensional on any scale.

This is specifically true in the non-perturbative regions of phase space where the entropic contributions to the effective action compete with those coming from the classical bare action, as explained above. In these regions it can easily happen that a type of configuration is entropically favoured that has no resemblance at all with

[8]Since we impose the kinematical constraint that the spatial volume at fixed t cannot become smaller than 5 tetrahedra – the minimal number required to build a simplicial manifold of topology S^3 – the volume never vanishes completely. More precisely, what we observe in addition to the bell-shaped part of the volume profile is the formation of a distinct "stalk" which is close to the minimal size of 5 everywhere.

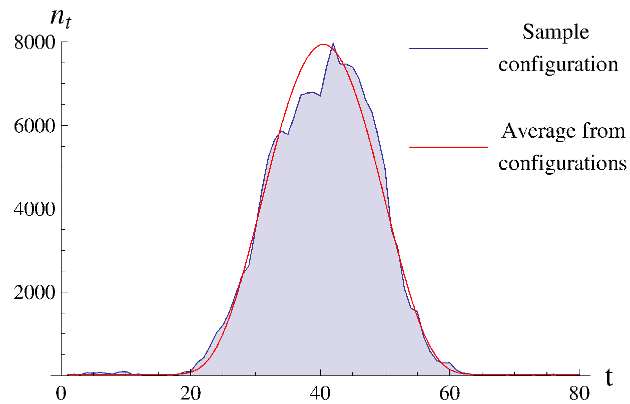

Figure 6: The three-volume of spatial slices as a function of proper time in phase C. Shown are a sample configuration of the volume profile, as well as the expectation value of the same quantity.

an extended four-dimensional universe. Just from looking at the volume profiles, it is obvious that something like this does indeed happen in phases A and B, which as a result do not appear to have any classical limit resembling general relativity [26]. However, even in phase C the observed quantum universe is truly an outcome of non-perturbative dynamics, not a consequence of the dominance of the classical action.[9]

The fact that the path-integral measure can play a crucial role in determining the non-perturbative dynamics was a main lesson learned already earlier in the context of four-dimensional DT quantum gravity. When one considers a path integral ensemble of geometries obtained from gluing four-dimensional equilateral Euclidean simplices, with the only constraint that the topology should be that of S^4, one ends up with a universe of vanishing linear extension and infinite Hausdorff dimension [30]. This makes the situation depicted in Fig. 6 all the more remarkable!

3.2 The effective action

However, the surprises do not stop here. The smooth curve in Fig. 6 represents the expectation value of the volume profile, that is, the average over path integral configurations measured in the Monte Carlo simulations. For N_4 sufficiently large this curve is very precisely fitted by the function

$$\langle N_3(i) \rangle \propto N_4^{3/4} \cos^3 \left(\frac{i}{s_0 N_4^{1/4}} \right), \tag{26}$$

[9]Since we are working in Euclidean signature, dominance of the classical action would be fatal for the path integral, because of the action's unboundedness from below. In phase C, this instability is cured by the entropy of "microstates" or, in other words, the path-integral measure [28, 29].

where i denotes (integer) lattice time, N_4 the total number of four-simplices and $N_3(i)$ the number of tetrahedra at time i [31, 32], and s_0 is a constant.[10]

Can the functional form of the expectation value found in (26) be obtained directly from an action principle? The answer is yes [28]. A long time ago, Hartle and Hawking explored a minisuperspace approach to quantum gravity , where all gravitational (field) degrees of freedom at a fixed time are represented by a single number, the so-called scale factor or, equivalently, the total three-volume of the universe.[11] Taking this classically reduced formulation as the starting point of the quantization, finding a quantum theory of gravity is reduced to a quantum mechanical problem in one variable, the scale factor $a(t)$ [33].

The volume profile (26) of the emergent extended universe found in phase C of CDT quantum gravity can be derived from an "effective" action for the three-volume, namely,

$$S_{eff} = \frac{1}{24\pi G} \int dt \left(\frac{\dot{V_3}^2(t)}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t) \right), \tag{27}$$

where t denotes proper time, k_2 is a numerical constant and λ is a Lagrange multiplier, not a cosmological constant, because the total four-volume V_4 is kept fixed in the simulations. Intriguingly, one obtains exactly the same expression (up to an overall sign) when plugging a spatially homogeneous and isotropic ansatz for the metric $g_{\mu\nu}(x)$ into the Euclidean Einstein-Hilbert action, and re-expressing the dependence on the scale factor in terms of the three-volume $V_3(t) \propto a^3(t)$. The solution to the equations of motion derived from (27) is the Euclidean de Sitter universe (a round four-sphere), which as a function of proper time t results in the $\cos^3(t/V_4^{1/4})$-dependence of eq. (26).

Despite the fact that they lead to very similar results for the dynamics of the scale factor, let us stress that conceptually there is a big difference between the ansatz of Hartle and Hawking, who simply assumed a minisuperspace reduction from the outset, and studying the effective dynamics of (the expectation value of) the scale factor in a full theory of quantum gravity, as we are doing. The only small but important reminder of the non-perturbative origin of the action (27) is its overall sign, which is opposite to that found in Euclidean cosmology. It can be attributed directly to "entropic" contributions to the effective action. The solutions to the equations of motion are of course not affected by this sign difference. A discretization of the effective action (27) has the functional form

$$S_{discr} = k_1 \sum_i \left(\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + \tilde{k}_2 N_3^{1/3}(i) - \tilde{\lambda} N_3(i) \right). \tag{28}$$

We have managed to reconstruct it in detail from the simulation data for the volume-volume correlator $\langle V_3(t) V_3(t') \rangle$, and have also shown that the quantum fluctuations

[10] The formula is of course not valid in the stalk, where $N_3(i) \approx 5$.

[11] This rather crude approximation is borrowed from standard cosmology, where homogeneity and isotropy are assumed to give a realistic description of our universe on the very largest scales.

around the de Sitter "background geometry" are well described by the action (28), yet another non-trivial result [32].

The same data have allowed us to relate the continuum coupling constant G in (27) to the constant k_1 in (28) according to

$$G = \frac{a^2}{k_1} \frac{\sqrt{C_4}\, s_0^2}{3\sqrt{6}}, \tag{29}$$

where a is the lattice spacing and C_4 is essentially the volume of a four-simplex (for lattice spacing $a = 1$), but depends weakly on the ratio between $N_4^{(1,4)}$ and $N_4^{(2,3)}$ (since the (4,1)- and (3,2)-simplices only have identical four-volumes when $\alpha = 1$). This ratio, as well as the value of the constant s_0, defined in eq. (26), depend on the choice of the bare coupling constants κ_0 and Δ in phase C.

Let us consider a typical choice for these couplings, $(\kappa_0, \Delta) = (2.2, 0.6)$, positioning us in the interior of phase C. At this point in phase space, we have measured k_1 and with the help of (29) expressed Newton's constant and the Planck length ℓ_P in terms of the lattice spacing, resulting in

$$G \approx 0.23a^2, \qquad \ell_P \equiv \sqrt{G} \approx 0.48a. \tag{30}$$

From the identification of spacetime with a Euclidean de Sitter universe we have that $V_4 = 8\pi^2 R^4/3 = C_4 N_4 a^4$, where C_4 is the same quantity that appeared in (29). For the range of four-volumes used in the simulations, $N_4 \in [45.000, 360.000]$, the linear size πR of the quantum de Sitter universes lies between 12 and 21 Planck lengths ℓ_P. The small size of our universes is compatible with the fact that the observed quantum fluctuations in the three-volume are quite substantial, as illustrated by Fig. 6 (see also Fig. 7). For larger universes, the volume fluctuations will quickly become irrelevant.

However, in order to investigate quantum properties of spacetime at Planckian and even sub-Planckian length scales, we want to do the opposite, namely, make the universes smaller and in this way increase the small-scale resolution of the simulations. How can we improve on (30) such that a single Planck length ℓ_P corresponds not to just half a lattice spacing, but to many lattice spacings a? From eqs. (29) and (30) it is clear that when k_1 goes to zero, ℓ_P can become much larger than a. The question is whether we can adjust k_1 to go to zero. Since k_1 depends on the bare coupling constants κ_0 and Δ, we have performed a scan of phase C to determine its qualitative behaviour [32]. Moving toward the A-C phase transition, k_1 is indeed decreasing, without going all the way to zero in the range of coupling constants scanned so far. Approaching the B-C phase transition is more difficult, because the system undergoes a second-order transition, and we observe a corresponding critical slowing-down. As far as we can tell from the numerical data at this stage, k_1 does *not* decrease when we approach this transition. However, as we will see in the next section, having k_1 go to a fixed value different from zero is actually the behaviour predicted at an ultraviolet second-order transition line, and therefore compatible with the continuum scenario we have appealed to earlier.

3.3 Making contact with asymptotic safety

Let us return to the renormalization group equation (2), which was formulated in terms of the dimensionless coupling constant $\tilde{G} = G\,E^2$. Now that we have a UV cut-off, the lattice link length a, we can instead form the dimensionless quantity $\hat{G} = G/a^2$. From (29) it can essentially be identified with the inverse of k_1, which we can measure. We can reformulate the renormalization group in terms of the new short-distance cut-off as

$$G(a) = a^2 \hat{G}(a), \qquad a\frac{\mathrm{d}\hat{G}}{\mathrm{d}a} = -\beta(\hat{G}), \qquad \beta(\hat{G}) = 2\hat{G} - c\hat{G}^2 + \cdots , \tag{31}$$

where c depends on the constant ω of eq. (2). Near the putative non-Gaussian UV fixed point $\hat{G}^*$, we can expand $\hat{G}$ and k_1 to lowest order in a according to

$$\hat{G}(a) = \hat{G}^* - K a^{\tilde{c}}, \quad k_1(a) = k_1^* + \tilde{K} a^{\tilde{c}}, \tag{32}$$

for some K, $\tilde{K}$, where the approach to the fixed point is governed by the exponent

$$\tilde{c} = -\beta'(\hat{G}^*). \tag{33}$$

As explained in Sec. 2.1, in standard lattice theory one would now relate the lattice spacing near the fixed point to the bare coupling constants with the help of some correlation length ξ. However, in four-dimensional quantum gravity we do not yet have a suitable correlation length at our disposal which could play this role.

In search of an alternative, let us first consider the equation $V_4 = N_4 a^4$, which defines the dimensionful continuum four-volume V_4 in terms of the number N_4 of four-simplices and the lattice spacing. If we could consider V_4 as fixed, we could replace the a-dependence of (32) by a N_4-dependence, with the advantage that N_4 is a parameter we can straightforwardly control. Re-expressing eq. (32) in terms of N_4 yields

$$k_1(N_4) = k_1^* - K' N_4^{-\tilde{c}/4}, \tag{34}$$

for some K'. Since we can measure k_1, we could determine the flow to the fixed point. The question is now which lattice measurements we should perform in order to make eq. (34) applicable. Increasing N_4 while *staying* at a specific point (κ_0, Δ) in phase C does *not* correspond to keeping V_4 fixed, because during this process the size of the quantum fluctuations in the three-volume decreases relative to the expectation value of the three-volume. (More precisely, we already know that the ratio goes to zero like $1/N_4^{1/4}$.) Conversely, if "physics" is to be constant, which includes a constant V_4, that same ratio should also remain constant.

We will use this observation as our definition for what we mean by a "path of constant physics". If we had a correlation length available, we could increase N_4 and simultaneously *change* the bare coupling constants in such a way that the ratio of the correlation length to the linear extension of the universe of volume N_4 (both in terms of lattice units) stayed constant. In the absence of a suitable correlation

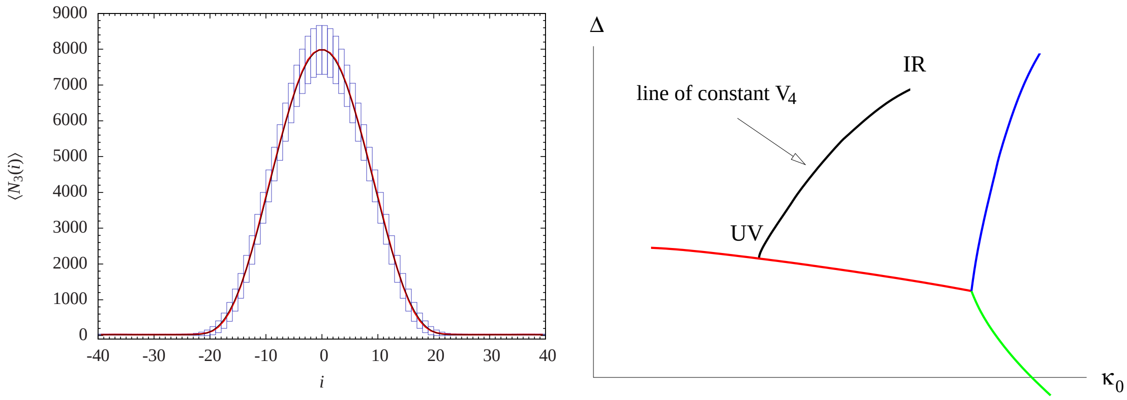

Figure 7: Left: Three-volume profile for given N_4, for specific values (κ_0, Δ) of the bare coupling constants. Also indicated is the magnitude of the three-volume fluctuations around the mean value. While the expectation value of the three-volume scales like $N_4^{3/4}$, the fluctuations only scale like $N_4^{1/2}$. Right: identifying a path of "constant physics" in the κ_0-Δ plane. Starting at some point in phase C, a path moving toward the UV phase transition is created by increasing N_4 and simultaneously adjusting κ_0 and Δ, such that the ratio of the size of the three-volume fluctuations and the expectation value of the three-volume remains constant.

length, we will use the magnitude of the three-volume fluctuations instead, and identify a "path of constant physics" as a trajectory in phase C along which the discrete four-volume N_4 grows, but the accompanying change in the bare couplings κ_0 and Δ ensures that the three-volume fluctuations likewise increase, in such a way that the ratio between the magnitude of the fluctuations and the mean three-volume stays the same. Fixing this ratio forces us to change bare coupling constants when we increase N_4, in this way tracing out a path that moves toward one of the phase transitions bordering phase C, see Fig. 7 (right) for a schematic illustration. Preliminary results from computer simulations to determine the flow defined in this way indicate that it should start quite close to the B-C phase transition if it should resemble the flow line of constant physics shown in the figure, raising again the issue of critical slowing-down near the B-C line.

4 Relation to Hořava-Lifshitz gravity

As described above, our CDT data in phase C can be fitted well to the functional form (28), which in turn can be seen as a discretized version of the minisuperspace action (27). There is a residual ambiguity in the interpretation of the discrete time coordinate appearing in the identification (26), which can be thought of as an overall, finite scaling between the time and spatial directions. As we have emphasized, due to the entropic nature of the effective action, there is no compelling reason to take the geometric length assignments of the regularized theory literally. We have identified

the time "coordinate" t with continuum proper time in such a way that we obtain a round four-sphere, which is a perfectly legitimate and physically well-motivated choice. However, as we vary the bare couplings κ_0 and Δ, the overall shape of the computer-generated universe changes in terms of the number of lattice spacings in the time direction relative to those in the spatial directions. Although this change is qualitatively in agreement with the change of α as a function of κ_0 and Δ, there is no detailed quantitative agreement.

Instead of choosing continuum time to be consistent with a continuum S^4-geometry as one moves in phase space, one may be able to find a modified action which describes the observed behaviour without performing an overall time rescaling which depends on κ_0 and Δ. This may be especially appropriate in the vicinity of the phase transition, where the length scales one is probing become increasingly Planckian, and one would expect significant contributions to the effective dynamics from terms not contained in the infrared form of the Einstein-Hilbert action including higher-order curvature terms.

We will consider yet another generalization, which suggests itself because of the built-in anisotropy between time and space of the CDT set-up, namely, a deformation à la Hořava-Lifshitz [34]. A corresponding effective Euclidean continuum action, including measure contributions, and expressed in terms of standard metric variables could be of the form

$$S_H = \frac{1}{16\pi G} \int \mathrm{d}^3x \ \mathrm{d}t \ N\sqrt{g}\Big((K_{ij}K^{ij} - \lambda K^2) + (-\gamma R^{(3)} + 2\Lambda + V(g_{ij})\Big), \qquad (35)$$

where K_{ij} denotes the extrinsic curvature and g_{ij} the three-metric of the spatial slices, $R^{(3)}$ the corresponding three-dimensional scalar curvature, N the lapse function, and finally $V(g_{ij})$ a "potential" which in Hořava's continuum formulation would contain higher orders of spatial derivatives, potentially rendering S_H renormalizable. In our case we are not committed to any particular choice of potential $V(g_{ij})$, since we are not imposing renormalizability of the theory in any conventional sense.

An effective $V(g_{ij})$ could be generated by entropy, i.e. by the measure, and may not relate to any discussion of the theory being renormalizable. The kinetic term depending on the extrinsic curvature is the most general such term which is at most second order in time derivatives and consistent with spatial diffeomorphism invariance. The parameter λ appears in the (generalized) DeWitt metric, which defines an ultralocal metric on the classical space of all three-metrics[12], and the parameter γ can be related to a relative scaling between time and spatial directions. Setting $\lambda = \gamma = 1$ and $V = 0$ in (35) we recover the standard (Euclidean) Einstein-Hilbert action.

[12]The value of λ governs the signature of the generalized DeWitt metric

$$G_\lambda^{ijkl} = \frac{1}{2}\sqrt{\det g}(g^{ik}g^{jl} + g^{il}g^{jk} - 2\lambda g^{ij}g^{kl}),$$

which is positive definite for $\lambda < 1/3$, indefinite for $\lambda = 1/3$ and negative definite for $\lambda > 1/3$. The role of λ in three-dimensional CDT quantum gravity has been analyzed in detail in [35].

Making a simple minisuperspace ansatz with compact spherical slices, which assumes homogeneity and isotropy of the spatial three-metric g_{ij}, and fixing the lapse to $N = 1$, the Euclidean action (35) becomes a function of the scale factor $a(t)$ (see also [36, 37, 38], as well as [39] for related work in 2+1 dimensions), that is,

$$S_{mini} = \frac{2\pi^2}{16\pi G} \int \mathrm{d}t\ a(t)^3 \Big(3(1-3\lambda)\ \frac{\dot{a}^2}{a^2} - \gamma\ \frac{6}{a^2} + 2\Lambda + \tilde{V}(a) \Big). \tag{36}$$

The first three terms in the parentheses define the IR limit (which in Hořava-Lifshitz gravity is assumed to include a flowing of λ to its "GR value"), while the potential term $\tilde{V}(a)$ contains inverse powers of the scale factor a coming from possible higher-order spatial derivative terms.

Our reconstruction of the effective action from the computer data is compatible with the functional form (36) of the minisuperspace action. If we were able to extract the constant $\tilde{k}_2$ in front of the potential term in (28), it would enable us to fix the ratio $(1-3\lambda)/2\gamma$ appearing in (36) [40]. At this stage, the precision of our measurements is insufficient to do so. The same is true for our attempts to determine $\tilde{V}(a)$ for small values of the scale factor, which is important for understanding UV quantum corrections to the potential near $a(t) = 0$. Once we have developed a better computer algorithm which allows us to approach the B-C phase transition line more closely, investigating such Planckian properties and testing scenarios of Hořava-Lifshitz type will be within reach.

4.1 Conclusions

In constructing a theory of quantum gravity using Causal Dynamical Triangulations, one of our initial inputs was the Regge action, which appears in the weights of individual spacetimes in the gravitational path integral. However, as we have emphasized repeatedly, the full effective action generated dynamically by performing the non-perturbative sum over histories is only indirectly related to this "bare" action. Likewise, the coupling constant k_1, which appears in front of the effective action and we view as related to the gravitational coupling constant G, has no obvious direct relation to the "bare" coupling κ_0 appearing in the Regge action.

Nevertheless, the leading terms in the effective action for the scale factor are precisely the ones present in (27) or, more generally, in the effective Hořava-Lifshitz action (36), at least for sufficiently large values of the scale factor. The fact that a kinetic term quadratic in derivatives appears as the leading term in the effective action is perhaps less surprising, but that the correct powers of the (undifferentiated) variable $N_3(i)$ appear in both the kinetic and potential terms in (28) is rather remarkable and very encouraging for the entire CDT quantization program.

For the range of bare coupling constants and four-volumes investigated until now our results are compatible with the Einstein-Hilbert action. Better data and more observables will be required to discriminate between a "pure gravity" behaviour and an anisotropic deformation à la Hořava-Lifshitz in the deep ultraviolet. A beautiful

feature of CDT quantum gravity is that entirely non-perturbative questions of this kind can be formulated explicitly and addressed with the non-perturbative lattice tools available, and – if one is lucky – be answered quantitatively.

Acknowledgment

JA and AG thank the Danish Research Council for financial support via the grant "Quantum gravity and the role of black holes", and the EU for support through the ERC Advanced Grant 291092, "Exploring the Quantum Universe" (EQU). JJ acknowledges a partial support of the International PhD Projects Programme of the Foundation for Polish Science within the European Regional Development Fund of the European Union, agreement no. MPD/2009/6. RL acknowledges support through several Projectruimte grants by the Dutch Foundation for Fundamental Research on Matter (FOM).

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